Analysis of Vibrations of Clustered Boosters

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The dynamic behavior of single-tank boosters has been studied extensively in connection with existing vehicles of comparatively small payloads. For future missions with larger payloads and distances of interplanetary magnitude, more powerful boosters need to be developed. It is believed that, in view of such missions, clustered boosters will come into the picture. In this paper an analysis for a four-tank booster is presented. An idealized booster is treated initially with uniformly distributed flexibility and mass. It is found that the torsional mode becomes important here in contrast to the case of the single-tank configuration; a strong coupling between the torsional and flexural modes of the tank occurs. For certain frequencies, simple modes are singled out. In certain designs, the frequencies corresponding to these modes might lie in the lower part of the spectrum. This leads to closed-form solutions and totally or partially uncoupled modes, a conclusion of extreme importance to the control problem. For an actual booster, where the flexibilities and masses are not uniformly distributed, a lumped-parameter technique is outlined briefly.

Introduction

IN the dynamics of a single-tank booster, the mode of vibration which is considered primarily is simple bending. The plane of vibration becomes immaterial because of the axial symmetry of the booster. It has been found in actual boosters that the torsional mode becomes unimportant, and that the modes are uncoupled.

There is no axial symmetry in the case of clustered boosters; therefore, the modes of vibration become more complicated. Bending modes are likely to occur in several different planes simultaneously while torsional modes are also occurring. Certain symmetries are expected in actual configurations, but instead of axial symmetry, one expects to find some planes of symmetry to which one can refer the various modes. For certain frequencies, it is possible to find uncoupled modes or simple combinations of independent modes. These cases are studied in this paper, since they are of practical importance.

In solving the problem, separate beam-type differential equations are considered for each tank. Since any shell-type mode of vibration is excluded, it is assumed that each cross section of a tank has two components (x direction and y direction) of rigid body translation and that each cross section of the four tanks considered as a unit has two components (x direction and y direction) of rigid-body translation. The tanks are assumed to be pin-joined at the points A, B, C, D, A', B , C , and D' as shown in Fig. 1. They are free to vibrate independently at all other points. It is obvious from the way the tanks are joined that the axis of rotation is the central z axis of the booster. The differential equations of motion then are formulated in terms of these displacements (x and y translations) considered as functions of z and the time t. After separating the time, one arrives at a system of ordinary differential equations. The characteristic determinant of this system is studied, and the number of real and imaginary roots is found. By applying the appropriate stress and displacement boundary conditions, one finds that the unknown constants of integration form a 40 × 40 determinant.

The characteristic determinant of this system is reduced to a 30 × 30 determinant by a process of elimination which brings out some interesting symmetries of the system of characteristic functions. Certain simple modes are discovered by studying some obvious double roots of the final determinant. To investigate the useful spectrum of frequencies, it generally is necessary to obtain the roots of the characteristic determinant by means of a digital computer.

The use of beam equations in this study needs some explanation. The dynamic interaction between liquid motions and elastic deformations of the wall of a single tank is an extremely complex problem. A full understanding of this behavior in single tank systems is necessary before applying the theory to a clustered configuration. However, in the design of single booster systems, beam theory has been quite successful in adequately predicting the lower frequency bending mode properties. This fact is understandable, since the sloshing modes generally are not coupled strongly to the shell modes because of their great frequency separation.

It is natural, therefore, to apply beam theory in the study of clustered tanks and thereby consider the new features in such a configuration.

1. Equations of Motion

1.1 Assumptions and Definitions

It is assumed here that the tanks are joined only at the points A, B, C, D, A', B', C', and D', as depicted in Fig. 1. Because of this connection, the end sections of the whole body have a rigid-body motion in the xy plane. Because the xz and yz planes are planes of symmetry, any rigid-body rotation of the whole booster will take place around the z axis. Thus, the rigid-body motion of an end section is described by three independent components, namely, translation in the x direction, translation in the y direction, and rotation around the z axis.

The z axis was chosen as the axis of rotation for each cross section of any tank since this choice does not cause any additional rigid-body motion (for small displacements) and it furnishes simple, kinematic boundary conditions. However, the flexural rotation of the end section of each tank in the xz and yz planes need not be equal because it was assumed that the points A, B, C, D, A', B', C', and D' are pin-joints. This assumption eliminates the need for introducing additional reaction bending moments transmitted through the points of connection between the tanks, and it simplifies the static boundary conditions. It is believed that this assumption is not too restrictive for an actual design.

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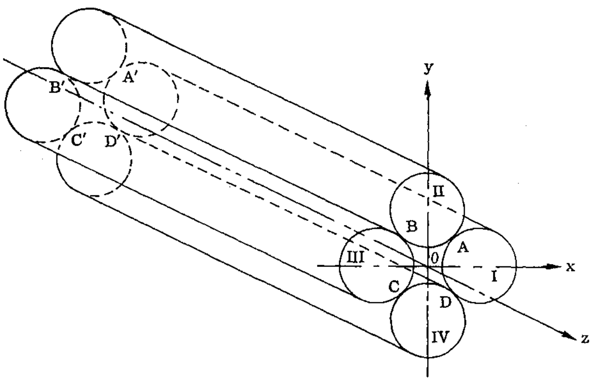

Fig. 1 Geometry of booster and axes of reference

Three independent components of motion are now defined for each cross section z of tank I:

$X_I(z,t)$ = displacement in the x direction
$y_I(z,t)$ = displacement in the y direction
$\theta_I(z,t)$ = rotation around the z axis

For convenience, the rotation is defined by the quantity

$$\phi_I(z,t) = 2^{1/2}R\theta_I(z,t) \tag{1}$$

where R equals the radius of each tank. For the other tanks, the same letters are used for the displacements and the appropriate subscript (I, II, III, or IV) in accordance with the labeling shown in Fig. 1.

1.2 Use of Existing Theory

Using the engineering theory of bending and torsion of beams, one can now write the following equations:

$$EI\frac{\partial^4 x_i(z,t)}{\partial z^4} = W_{xi} \qquad EI\frac{\partial^4 y_i(z,t)}{\partial z^4} = W_{yi}$$

$$\frac{GC}{2^{1/2}R}\frac{\partial^2\phi_i(z,t)}{\partial z^2} = -W\phi_i \tag{2}$$

where

i = I, II, III, or IV
EI = uniform flexural stiffness of each beam
GC = uniform torsional stiffness of each beam
W_{xi}, W_{yi} = transverse loads per unit length in the x and y directions
$W\phi_i$ = torque distributed per unit length

For a free vibration, these loads are inertial loads, i.e., proportional to the accelerations.

By considering the various components of acceleration (Fig. 2) for a mass element dm having coordinates ξ and η with respect to the x and y axes, respectively, and by summing up all the elementary inertial forces over the entire cross section of each beam, one finds the loads W_{xi}, W_{yi}, and $W\phi_i$ per unit length. If

W = weight of each tank per unit length (assumed here as constant along the length)
J_{Pt}, J_{Pl} = moment of inertia of the cross section of the shell tank and the cross section of the liquid mass, respectively
γ_t, γ_l = specific weights of the tank and liquid, respectively

then the torque load of tank I is

$$W\phi_I = \frac{1}{2^{1/2}g}\left\{\frac{J_{Pl}\gamma_l}{R} + \frac{J_{Pt}\gamma_t}{R} + 2RW\right\}\frac{\partial^2\phi_I}{\partial t^2} - \frac{W}{g}R(2)^{1/2}\frac{\partial^2 y_I}{\partial t^2} \tag{3}$$

It is easy to prove, by considering the geometry of the cross sections of the tank and the liquid, that the first term in Eq. (3) is equal to

$$(R/2^{1/2}g)[(5+m)/2]W \tag{4}$$‡

where m equals the ratio of the weight of the shell tank to the total weight of the shell plus liquid.

Using Eq. (4) and summing the inertial forces of the mass elements over the entire cross section, one finds for tank I

$$W_{xI} = -\frac{W}{g}\frac{\partial^2 x_I}{\partial t^2}$$

$$W_{yI} = -\frac{W}{g}\frac{\partial^2}{\partial t^2}(y_I + \phi_I) \tag{5}$$

$$W\phi_I = -\frac{R}{2(2)^{1/2}g}(5+m)W\frac{\partial^2\phi_I}{\partial t^2} - \frac{W}{g}R(2)^{1/2}\frac{\partial^2 y_I}{\partial t^2}$$

for tank II

$$W_{xII} = -\frac{W}{g}\frac{\partial^2}{\partial t^2}(x_{II} - \phi_{II})$$

$$W_{yII} = -\frac{W}{g}\frac{\partial^2 y_{II}}{\partial t^2} \tag{6}$$

$$W\phi_{II} = -\frac{R}{2(2)^{1/2}g}(5+m)W\frac{\partial^2\phi_{II}}{\partial t^2} + \frac{W}{g}R(2)^{1/2}\frac{\partial^2 x_{II}}{\partial t^2}$$

for tank III

$$W_{xIII} = -\frac{W}{g}\frac{\partial^2 x_{III}}{\partial t^2}$$

$$W_{yIII} = -\frac{W}{g}\frac{\partial^2}{\partial t^2}(y_{III} - \phi_{III}) \tag{7}$$

$$W\phi_{III} = -\frac{R}{2(2)^{1/2}g}(5+m)W\frac{\partial^2\phi_{III}}{\partial t^2} + \frac{W}{g}R(2)^{1/2}\frac{\partial^2 y_{III}}{\partial t^2}$$

for tank IV

$$W_{xIV} = -\frac{W}{g}\frac{\partial^2}{\partial t^2}(x_{IV} + \phi_{IV})$$

$$W_{yIV} = -\frac{W}{g}\frac{\partial^2 y_{IV}}{\partial t^2} \tag{8}$$

$$W\phi_{IV} = -\frac{R}{2(2)^{1/2}g}(5+m)W\frac{\partial^2\phi_{IV}}{\partial t^2} - \frac{W}{g}R(2)^{1/2}\frac{\partial^2 x_{IV}}{\partial t^2}$$

By substituting the inertial forces from Eqs. (5–8) into Eqs. (2), one finds the elasto-kinetic equations of the booster. By substituting the parameters

$$a = (W/gEI)^{1/2} \qquad b = (EI/GC) \tag{9}$$

the elasto-kinetic equations take the following form: for tank I

$$\frac{\partial^4 x_I}{\partial z^4} + a^2\frac{\partial^2 x_I}{\partial t^2} = 0$$

$$\frac{\partial^4 y_I}{\partial z^4} + a^2\frac{\partial^2 y_I}{\partial t^2} + a^2\frac{\partial^2\phi_I}{\partial t^2} = 0 \tag{10}$$

$$\frac{\partial^2\phi_I}{\partial z^2} - b\left(\frac{5+m}{2}\right)R^2a^2\frac{\partial^2\phi_I}{\partial t^2} - 2bR^2a^2\frac{\partial^2 y_I}{\partial t^2} = 0$$

‡ For a clean internal design, the shell can rotate around the liquid while the latter remains at rest. In this case, $J_{pl} = 0$ and the factor $(5+m)/2$ in Eq. (4) should be substituted by $(2+m)$. It can be proved that Eq. (30) has one positive and two negative real roots for this case. This adjustment, therefore, does not invalidate the present solution.

for tank II

$$\frac{\partial^4 x_{II}}{\partial z^4} + a^2 \frac{\partial^2 x_{II}}{\partial t^2} - a^2 \frac{\partial^2 \phi_{II}}{\partial t^2} = 0$$
$$\frac{\partial^4 y_{II}}{\partial z^4} + a^2 \frac{\partial^2 y_{II}}{\partial t^2} = 0 \tag{11}$$
$$\frac{\partial^2 \phi_{II}}{\partial t^2} - b\left(\frac{5+m}{2}\right) R^2 a^2 \frac{\partial^2 \phi_{II}}{\partial t^2} + 2bR^2 a^2 \frac{\partial^2 x_{II}}{\partial t^2} = 0$$

for tank III

$$\frac{\partial^4 x_{III}}{\partial z^4} + a^2 \frac{\partial^2 x_{III}}{\partial t^2} = 0$$
$$\frac{\partial^4 y_{III}}{\partial z^4} + a^2 \frac{\partial^2 y_{III}}{\partial t^2} - a^2 \frac{\partial^2 \phi_{III}}{\partial t^2} = 0 \tag{12}$$
$$\frac{\partial^2 \phi_{III}}{\partial z^2} - b\left(\frac{5+m}{2}\right) R^2 a^2 \frac{\partial^2 \phi_{III}}{\partial t^2} + 2bR^2 a^2 \frac{\partial^2 y_{III}}{\partial t^2} = 0$$

for tank IV

$$\frac{\partial^4 x_{IV}}{\partial z^4} + a^2 \frac{\partial^2 x_{IV}}{\partial t^2} + a^2 \frac{\partial^2 \phi_{IV}}{\partial t^2} = 0$$
$$\frac{\partial^4 y_{IV}}{\partial z^4} + a^2 \frac{\partial^2 y_{IV}}{\partial t^2} = 0 \tag{13}$$
$$\frac{\partial^2 \phi_{IV}}{\partial z^2} - b\left(\frac{5+m}{2}\right) R^2 a^2 \frac{\partial^2 \phi_{IV}}{\partial t^2} - 2bR^2 a^2 \frac{\partial^2 x_{IV}}{\partial t^2} = 0$$

2. Eigenfunctions of Vibrations

2.1 Method of Solution

The solution of the system of partial differential equations (10–13) is obtained by the method of separation of variables. For the function $x_I(z,t)$, for example, one can write

$$x_I(z,t) = X_I(z)\, T(t) \tag{14}$$

Substituting Eq. (14) into Eqs. (10–13), one finds

$$x_I(z,t) = X_I(z)\,[K \cos\omega t + L \sin\omega t] \tag{15}$$

etc., where ω is the frequency of vibration. (In the separation of variables process, the characteristic function of each displacement is denoted by the corresponding capital letter, with a subscript denoting the label of the tank according to Fig. 1.) The characteristic functions (eigenfunctions), which give the shape of the mode of vibration obtained from these substitutions, are solutions of the following system of ordinary differential equations: for tank I

$$\frac{d^4 X_I}{dz^4} - a^2\omega^2 X_I = 0$$
$$\frac{d^4 Y_I}{dz^4} - a^2\omega^2 Y_I - a^2\omega^2 \Phi_I = 0 \tag{16}$$
$$\frac{d^2 \Phi_I}{dz^2} + b\left(\frac{5+m}{2}\right) R^2 a^2\omega^2 \Phi_I + 2bR^2 a^2 \omega^2 Y_I = 0$$

for tank II

$$\frac{d^4 X_{II}}{dz^4} - a^2\omega^2 X_{II} + a^2\omega^2 \Phi_{II} = 0$$
$$\frac{d^4 Y_{II}}{dz^4} - a^2\omega^2 Y_{II} = 0 \tag{17}$$
$$\frac{d^2 \Phi_{II}}{dz^2} + b\left(\frac{5+m}{2}\right) R^2 a^2\omega^2 \Phi_{II} - 2bR^2 a^2 \omega^2 X_{II} = 0$$

for tank III

$$\frac{d^4 X_{III}}{dz^4} - a^2\omega^2 X_{III} = 0$$
$$\frac{d^4 Y_{III}}{dz^4} - a^2\omega^2 Y_{III} + a^2\omega^2 \Phi_{III} = 0 \tag{18}$$
$$\frac{d^2 \Phi_{III}}{dz^2} + b\left(\frac{5+m}{2}\right) R^2 a^2\omega^2 \Phi_{III} - 2bR^2 a^2 \omega^2 Y_{III} = 0$$

for tank IV

$$\frac{d^4 X_{IV}}{dz^4} - a^2\omega^2 X_{IV} - a^2\omega^2 \Phi_{IV} = 0$$
$$\frac{d^4 Y_{IV}}{dz^4} - a^2\omega^2 Y_{IV} = 0 \tag{19}$$
$$\frac{d^2 \Phi_{IV}}{dz^2} + b\left(\frac{5+m}{2}\right) R^2 a^2\omega^2 \Phi_{IV} + 2bR^2 a^2 \omega^2 X_{IV} = 0$$

A study of the foregoing equations indicates that in each tank the vibration in one direction (x or y) is uncoupled, whereas the vibration in the other direction is coupled with the torsional vibration. Since the equation of the uncoupled vibration and the two equations of the coupled vibrations are similar, apart from interchanges of the sign, the solutions for all the tanks will be similar. For example, if one makes the substitutions $X_I \rightarrow Y_{II}$, $Y_I \rightarrow X_{II}$, $\Phi_I \rightarrow -\Phi_{II}$ in the system for the first tank, Eqs. (16), one finds a system similar to that of Eqs. (17). Such similarities also exist for the other two tanks.

2.2 Investigation of the Characteristic System

The characteristic system for each tank is similar to the following system:

$$\frac{d^4 f(z)}{dz^4} - a^2\omega^2 f(z) = 0 \tag{20}$$

$$\frac{d^4 g(z)}{dz^4} - a^2\omega^2 g(z) - a^2\omega^2 h(z) = 0 \tag{21}$$

$$\frac{d^2 h(z)}{dz^2} + b\left(\frac{5+m}{2}\right) R^2 a^2\omega^2 h(z) + 2bR^2 a^2\omega^2 g(z) = 0 \tag{22}$$

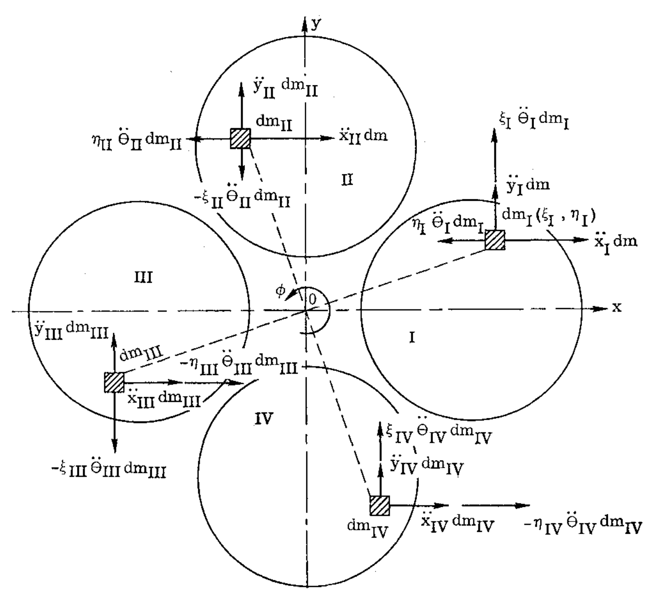

Fig. 2 Acceleration components

The first equation has the general solution

$$f(z) = A_1 \cosh\frac{\lambda_1}{R} z + A_2 \sinh\frac{\lambda_1}{R} z + A_3 \cos\frac{\lambda_1}{R} z + A_4 \sin\frac{\lambda_1}{R} z \quad (23)$$

where

$$\lambda_1 = R\,(a\omega)^{1/2} \quad (24)$$

To find a solution of the coupled system, Eqs. (21) and (22), one discovers that particular solutions of the form

$$g(z) = Be^{(\lambda/R)z} \qquad h(z) = Ce^{(\lambda/R)z} \quad (25)$$

where λ is, in general, complex, and B and C are constants, will satisfy this system if B and C are solutions of the linear homogeneous system:

$$\begin{gathered}[(\lambda/R)^4 - a^2\omega^2]\,B - a^2\omega^2 C = 0 \\ 2bR^2a^2\omega^2 B + \{(\lambda/R)^2 + b[(5+m)/2]\,R^2a^2\omega^2\}\,C = 0\end{gathered} \quad (26)$$

This system has a nontrivial solution if the value of the determinant of the coefficients of B and C is zero. In such a case, one can solve Eqs. (26) only for the ratio B/C. The determinant of Eqs. (26) is

$$\begin{bmatrix} \left[\left(\dfrac{\lambda}{R}\right)^4 - a^2\,\omega^2\right] & -\,a^2\,\omega^2 \\ 2bR^2a^2\omega^2 & \left[\left(\dfrac{\lambda}{R}\right)\right]^2 + b\left(\dfrac{5+m}{2}\right) R^2a^2\omega^2 \end{bmatrix} = 0 \quad (27)$$

By expanding this determinant, one obtains

$$\lambda^6 + b\left(\frac{5+m}{2}\right) P\lambda^4 - P\lambda^2 - b\left(\frac{1+m}{2}\right) p^2 = 0 \quad (28)$$

where

$$P = R^4a^2\omega^2 \quad (29)$$

[Note from Eq. (24) that $\lambda_1 = 4(P)^{1/2}$.] Therefore, the exponent λ in Eqs. (25) can take the value of one of the six roots of Eq. (28). A linear combination of six terms of the form $B_ie^{(\lambda_i/R)z}$ and $C_ie^{(\lambda_i/R)z}$, where $i = 1, 2, \ldots 6$, will be the general solution of Eqs. (21) and (22).

2.3 Real and Imaginary Roots of the Characteristic System

To exclude complex solutions, one needs to know which roots of Eq. (28) are real and which are imaginary. Therefore, it is expedient to investigate the roots of Eq. (28). Since Eq. (28) is a function of λ^2, by substituting $\lambda^2 = \psi$, one obtains

$$\psi^3 + b\left(\frac{5+m}{2}\right) P\psi^2 - P\psi - b\left(\frac{1+m}{2}\right) P^2 = 0 \quad (30)$$

The form of the roots of a cubic equation

$$\psi^3 + c\psi^2 + d\psi + e = 0 \quad (31)$$

depends on sign of the quantity

$$l = q^2 + v^3 \quad (32)$$

where

$$q = (c^3/27) - \tfrac{1}{6}\,cd + \tfrac{1}{2}\,e \quad (33)$$

$$v = (d/3) - (c^2/9) \quad (34)$$

If $q^2 + v^3 > 0$, the cubic, Eq. (31), has one real root and two complex conjugates. If $q^2 + v^3 < 0$, the equation has three real roots. By comparing Eq. (30) with Eqs. (31–34), one finds

$$q^2 + v^3 = -\frac{P^3b^2}{27}\left\{ b^2 \left(\frac{5+m}{2}\right)^3\left(\frac{1+m}{2}\right) P^2 + \frac{1}{2}\,(11 + 8m - m^2)\,P + 1\right\} \quad (35)$$

Since $m < 1$, it follows that *all the roots of Eq. (30) are real.* It is further necessary to know the sign of the roots of Eq. (30). This is because $\lambda_i = \pm\,\psi_i^{1/2}$, where $i = 1, 2, 3$, and one needs to know whether λ_i is real or imaginary. To that effect, Routh's rule is applied (see Ref. 4). For a cubic polynomial

$$a_0\psi^3 + a_1\psi^2 + a_2\psi + a_3 = 0 \quad (36)$$

one forms the following Routh's table:

$$\begin{array}{l|ll} \psi^3 \rightarrow & a_0 & a_2 \\ \psi^2 \rightarrow & a_1 & a_3 \\ \psi^1 \rightarrow & \dfrac{a_1a_2 - a_0a_3}{a_1} & 0 \\ \psi^0 \rightarrow & a_3 & 0 \end{array} \quad (37)$$

The number of positive roots of Eq. (36) is equal to the number of changes of sign in the left-hand column of the foregoing table. Comparing Eqs. (36) and (37) with Eq. (30), one forms the following Routh's table:

$$\begin{array}{l|ll} \psi^3 \rightarrow & 1 & -\,P \\ \psi^2 \rightarrow & b\left(\dfrac{5+m}{2}\right) P & -\,b\left(\dfrac{1+m}{2}\right) p^2 \\ \psi^1 \rightarrow & -\,2bP^2 & 0 \\ \psi^0 \rightarrow & -\,b\left(\dfrac{1+m}{2}\right) P^2 & 0 \end{array} \quad (38)$$

One notices that the left-hand column changes sign once Therefore, *Eq. (30) has one positive root and two negative roots for positive values of parameters P and b of the present problem.* Let the positive root of Eq. (30) be called ψ_2 and the two negative roots ψ_3 and ψ_4. The six roots of Eq. (28) can be written as $\pm\lambda_2$, $\pm i\lambda_3$, and $\pm i\lambda_4$, where

$$\begin{aligned}\lambda_2 &= \psi_2^{1/2} \\ \lambda_3 &= |\psi_3|^{1/2} \\ \lambda_4 &= |\psi_4|^{1/2}\end{aligned} \quad (39)$$

2.4 General Solution

For any positive value of the parameters P and b (only positive values have physical meaning here), the general solutions of system Eqs. (21) and (22) are

$$g(z) = A_5 \cosh\frac{\lambda_2}{R} z + A_6 \sinh\frac{\lambda_2}{R} z + A_7 \cos\frac{\lambda_3}{R} z + A_8 \sin\frac{\lambda_3}{R} z + A_9 \cos\frac{\lambda_4}{R} z + A_{10} \sin\frac{\lambda_4}{R} z \quad (40)$$

$$h(z) = K_2\left(A_5 \cosh\frac{\lambda_2}{R} z + A_6 \sinh\frac{\lambda_2}{R} z\right) + K_3\left(A_7 \cos\frac{\lambda_3}{R} z + A_8 \sin\frac{\lambda_3}{R} z\right) + K_4\left(A_9 \cos\frac{\lambda_4}{R} z + A_{10} \sin\frac{\lambda_4}{R} z\right) \quad (41)$$

where $A_5, A_6, \ldots A_{10}$ are arbitrary constants, and

$$K_i = [(\lambda_i^4/P) - 1] \quad (42)$$

where $i = 2, 3$, and 4. Applying Eqs. (23, 40, and 41) to each individual tank in accordance to system Eqs. (16–19), one finds that the eigenfunctions are given by the following: for tank I

$$X_{\rm I} = A_1^{\rm I}\cosh\frac{\lambda_1}{R}z + A_2^{\rm I}\sinh\frac{\lambda_1}{R}z + A_3^{\rm I}\cos\frac{\lambda_1}{R}z + A_4^{\rm I}\sin\frac{\lambda_1}{R}z$$

$$Y_{\rm I} = A_5^{\rm I}\cosh\frac{\lambda_2}{R}z + A_6^{\rm I}\sinh\frac{\lambda_2}{R}z + A_7^{\rm I}\cos\frac{\lambda_3}{R}z + A_8^{\rm I}\sin\frac{\lambda_3}{R}z + A_9^{\rm I}\cos\frac{\lambda_4}{R}z + A_{10}^{\rm I}\sin\frac{\lambda_4}{R}z \quad (43)$$

$$\Phi_{\rm I} = K_2\left(A_5^{\rm I}\cosh\frac{\lambda_2}{R}z + A_6^{\rm I}\sinh\frac{\lambda_2}{R}z\right) + K_3\left(A_7^{\rm I}\cos\frac{\lambda_3}{R}z + A_8^{\rm I}\sin\frac{\lambda_3}{R}z\right) + K_4\left(A_9^{\rm I}\cos\frac{\lambda_3}{R}z + A_{10}^{\rm I}\sin\frac{\lambda_4}{R}z\right)$$

for tank II

$$X_{\rm II} = A_5^{\rm II}\cosh\frac{\lambda_2}{R}z + A_6^{\rm II}\sinh\frac{\lambda_2}{R}z + A_7^{\rm II}\cos\frac{\lambda_3}{R}z + A_8^{\rm II}\sin\frac{\lambda_3}{R}z + A_9^{\rm II}\cos\frac{\lambda_4}{R}z + A_{10}^{\rm II}\sin\frac{\lambda_4}{R}z$$

$$Y_{\rm II} = A_1^{\rm II}\cosh\frac{\lambda_1}{R}z + A_2^{\rm II}\sinh\frac{\lambda_1}{R}z + A_3^{\rm II}\cos\frac{\lambda_1}{R}z + A_4^{\rm II}\sin\frac{\lambda_1}{R}z \quad (44)$$

$$\Phi_{\rm II} = -K_2\left(A_5^{\rm I}\cosh\frac{\lambda_2}{R}z + A_6^{\rm II}\sinh\frac{\lambda_2}{R}z\right) - K_3\left(A_7^{\rm I}\cos\frac{\lambda_3}{R}z + A_8^{\rm I}\sin\frac{\lambda_3}{R}z\right) - K_4\left(A_9^{\rm I}\cos\frac{\lambda_4}{R}z + A_{10}^{\rm I}\sin\frac{\lambda_4}{R}z\right)$$

for tank III

$$X_{\rm III} = A_1^{\rm III}\cosh\frac{\lambda_1}{R}z + A_2^{\rm III}\sinh\frac{\lambda_1}{R}z + A_3^{\rm III}\cos\frac{\lambda_1}{R}z + A_4^{\rm III}\sin\frac{\lambda_1}{R}z$$

$$Y_{\rm III} = A_5^{\rm III}\cosh\frac{\lambda_2}{R}z + A_6^{\rm III}\sinh\frac{\lambda_2}{R}z + A_7^{\rm III}\cos\frac{\lambda_3}{R}z + A_8^{\rm III}\sin\frac{\lambda_3}{R}z + A_9^{\rm III}\cos\frac{\lambda_4}{R}z + A_{10}^{\rm IV}\sin\frac{\lambda_4}{R}z \quad (45)$$

$$\Phi_{\rm III} = -K_2\left(A_6^{\rm III}\cosh\frac{\lambda_2}{R}z + A_7^{\rm III}\sinh\frac{\lambda_2}{R}z\right) - K_3\left(A_8^{\rm III}\cos\frac{\lambda_3}{R}z + A_9^{\rm III}\sin\frac{\lambda_3}{R}z\right) - K_4\left(A_9^{\rm III}\cos\frac{\lambda_4}{R}z + A_{10}^{\rm III}\sin\frac{\lambda_4}{R}z\right)$$

for tank IV

$$X_{\rm IV} = A_5^{\rm IV}\cosh\frac{\lambda_2}{R}z + A_6^{\rm IV}\sinh\frac{\lambda_2}{R}z + A_7^{\rm IV}\cos\frac{\lambda_3}{R}z + A_8^{\rm IV}\sin\frac{\lambda_3}{R}z + A_9^{\rm IV}\cos\frac{\lambda_4}{R}z + A_{10}^{\rm IV}\sin\frac{\lambda_4}{R}z$$

$$Y_{\rm IV} = A_1^{\rm IV}\cosh\frac{\lambda_1}{R}z + A_2^{\rm IV}\sinh\frac{\lambda_1}{R}z + A_3^{\rm IV}\cos\frac{\lambda_1}{R}z + A_4^{\rm IV}\sin\frac{\lambda_1}{R}z \quad (46)$$

$$\Phi_{\rm IV} = K_2\left(A_5^{\rm IV}\cosh\frac{\lambda_2}{R}z + A_6^{\rm IV}\sinh\frac{\lambda_2}{R}z\right) + K_3\left(A_7^{\rm IV}\cos\frac{\lambda_3}{R}z + A_8^{\rm IV}\sin\frac{\lambda_3}{R}z\right) + K_4\left(A_9^{\rm IV}\cos\frac{\lambda_4}{R}z + A_{10}^{\rm IV}\sin\frac{\lambda_4}{R}z\right)$$

where $A_1^{\rm I} \ldots A_{10}^{\rm IV}$ are constants to be determined by the boundary conditions.

3. Development of the Linear Homogeneous System

3.1 Boundary Conditions

One observes from Eqs. (43–46) that there are ten unknown constants of integration for each tank. Therefore, one has 40 unknowns and should seek 40 boundary conditions. These conditions are:

1) The x displacement of both end-sections must be the same for all tanks, i.e., $X_{\rm I} = X_{\rm II} = X_{\rm III} = X_{\rm IV}$ for $z = 0$ and $z = l$, where l = the length of the booster. This condition provides six equations.

2) The y displacement of both end-sections must be the same for all tanks, i.e., $Y_{\rm I} = Y_{\rm II} = Y_{\rm III} = Y_{\rm IV}$ for $z = 0$ and $z = l$. This condition furnishes six equations.

3) The twists of both end-sections must be the same for all tanks, i.e., $\Phi_{\rm I} = \Phi_{\rm II} = \Phi_{\rm III} = \phi_{\rm IV}$ for $z = 0$ and $z = l$. This condition furnishes six equations.

4) The total shear force in the x direction at each end must be zero. One knows from the engineering theory of bending that the shear force in a beam is proportional to the third derivative of the transverse displacement with respect to the axial coordinate. Therefore, this condition is satisfied if $\sum_{i={\rm I}}^{\rm IV} d^3X_i/dz^3 = 0$ for $z = 0$ and $z = l$. This condition furnishes two equations. The total shear force in the y direction at each end also must be zero, i.e., $\sum_{i={\rm I}}^{\rm IV} d^3Y_i/dz^3 = 0$ and $z = l$. This condition furnishes two equations.

5) The total torque at both ends is zero. Since the torque of a beam is proportional to the first derivative of the angle of twist, this condition leads to $\sum_{i={\rm I}}^{\rm IV} d\Phi_i/dz = 0$ for $z = 0$ and $z = l$, which furnishes two equations.

6) It is assumed here that each tank is simply supported at both ends on the two adjacent tanks. The bending moment in a certain direction is proportional to the second derivative of the transverse displacement in the corresponding direction with respect to the axial coordinates. This condition leads to the following 16 equations: $d^2X_i/dz^2 = 0$ for $z = 0$ and $z = l$, $d^2Y_i/dz^2 = 0$ for $z = 0$ and $z = l$, where i = I, II, III, and IV.

By counting the foregoing conditions, one observes that there are exactly 40 equations for the 40 unknowns. Applying these conditions into the solutions given by Eqs. (43–46) will result in a linear homogeneous system.

3.2 Mode Shapes

The system of Eqs. (48–79) will have a nontrivial solution if the determinant of the coefficients of the unknowns is zero. The elements of this determinant are functions of the parameter $P = R^4a^2\omega^2$. Therefore, the zeros of the determinant corresponds to the natural frequencies.

One can reduce the size of the determinant by eliminating some of the unknowns. This elimination process leads to some interesting conclusions as to the shape of the modes. Thus it can be shown that

$$A_1^{\mathrm{I}} = A_3^{\mathrm{I}} = A_1^{\mathrm{III}} = A_3^{\mathrm{III}} \tag{47}$$

Also

$$(A_2^{\mathrm{I}} - A_2^{\mathrm{III}}) \sinh\mu_1 + (A_4^{\mathrm{I}} - A_4^{\mathrm{III}}) \sin\mu_1 = 0 \tag{48}$$

and

$$(A_2^{\mathrm{I}} - A_2^{\mathrm{III}}) \sinh\mu_1 - (A_4^{\mathrm{I}} - A_4^{\mathrm{III}}) \sin\mu_1 = 0 \tag{49}$$

where $\mu_i = (l/R)\lambda_i$.

Eqs. (48) and (49) have a nonzero solution for the terms in the brackets if $(\sinh\mu_1)(\sin\mu_1) = 0$ or

$$\mu_1 = k\pi \tag{50}$$

where $k = 0, 1, 2, \ldots$. The value $\mu_1 = 0$ corresponds to $\omega = 0$, which is of no interest. The values of ω for $k \neq 0$ correspond to independent pure bending of each tank and will be discussed later. To find other roots of the determinant, one assumes $\mu_1 \neq k\pi$. From Eqs. (48) and (49), it follows that

$$A_2^{\mathrm{I}} = A_2^{\mathrm{III}} \qquad A_4^{\mathrm{I}} = A_4^{\mathrm{III}} \tag{51}$$

It can also be shown that

$$A_1^{\mathrm{II}} = A_3^{\mathrm{II}} = A_1^{\mathrm{IV}} = A_3^{\mathrm{IV}} \tag{52}$$

and

$$(A_2^{\mathrm{II}} - A_2^{\mathrm{IV}}) \sinh\mu_1 + (A_4^{\mathrm{II}} - A_4^{\mathrm{IV}}) \sin\mu_1 = 0 \tag{53}$$

$$(A_2^{\mathrm{II}} - A_2^{\mathrm{IV}}) \sinh\mu_1 - (A_4^{\mathrm{II}} - A_4^{\mathrm{IV}}) \sin\mu_1 = 0 \tag{54}$$

From Eqs. (53) and (54), by excluding the eigenfrequency $\mu_1 = k\pi$, one obtains

$$A_2^{\mathrm{II}} = A_2^{\mathrm{IV}} \qquad A_4^{\mathrm{II}} = A_4^{\mathrm{IV}} \tag{55}$$

Eqs. (47, 51, 52, and 55) now are substituted for 10 equations of the original 40×40 system. The 10 unknowns, A_3^{I}, A_1^{III}, A_3^{III}, A_3^{II}, A_1^{IV}, A_3^{IV}, A_2^{III}, A_2^{IV}, A_4^{III}, and A_4^{IV}, can be determined in terms of A_1^{I}, A_1^{II}, A_2^{I}, A_2^{II}, A_4^{I}, and A_4^{II}. Therefore, in the original system, only the linearly independent equations are retained after the foregoing substitution is made. By this process, the original system has been reduced to one of 30 equations with 30 unknowns. In addition to reducing the size of the characteristic determinant, from which the eigenfrequencies are calculated, this elimination has revealed the important conclusion that the following modes are equal:

$$X_{\mathrm{I}}(z) = X_{\mathrm{III}}(z) \qquad Y_{\mathrm{II}}(z) = Y_{\mathrm{IV}}(z) \tag{56}$$

Applying the foregoing substitution into Eqs. (43–46), the modes assume the following form: for tank I

$$X_{\mathrm{I}}(z) = A_1^{\mathrm{I}}\left(\cosh\frac{\lambda_1}{R}z + \cos\frac{\lambda_1}{R}z\right) + A_2^{\mathrm{I}} \sinh\frac{\lambda_1}{R}z + A_4^{\mathrm{I}} \sin\frac{\lambda_1}{R}z$$

$$Y_{\mathrm{I}}(z) = A_5^{\mathrm{I}} \cosh\frac{\lambda_2}{R}z + A_6^{\mathrm{I}} \sinh\frac{\lambda_2}{R}z + A_i^{\mathrm{I}} \cos\frac{\lambda_3}{R}z + A_8^{\mathrm{I}} \sin\frac{\lambda_3}{R}z + A_9^{\mathrm{I}} \cos\frac{\lambda_4}{R}z + A_{10}^{\mathrm{I}} \sin\frac{\lambda_4}{R}z \tag{57}$$

$$\Phi_{\mathrm{I}}(z) = K_2\left(A_5^{\mathrm{I}} \cosh\frac{\lambda_2}{R}z + A_6^{\mathrm{I}} \sinh\frac{\lambda_2}{R}z\right) + K_3\left(A_7^{\mathrm{I}} \cos\frac{\lambda_3}{R}z + A_8^{\mathrm{I}} \sin\frac{\lambda_3}{R}z\right) + K_4\left(A_9^{\mathrm{I}} \cos\frac{\lambda_4}{R}z + A_{10}^{\mathrm{I}} \sin\frac{\lambda_4}{R}z\right)$$

For tank II

$$X_{\mathrm{II}}(z) = A_5^{\mathrm{II}} \cosh\frac{\lambda_2}{R}z + A_6^{\mathrm{II}} \sinh\frac{\lambda_2}{R}z + A_7^{\mathrm{II}} \cos\frac{\lambda_3}{R}z + A_8^{\mathrm{II}} \sin\frac{\lambda_3}{R}z + A_9^{\mathrm{II}} \cos\frac{\lambda_4}{R}z + A_{10}^{\mathrm{II}} \sin\frac{\lambda_4}{R}$$

$$Y_{\mathrm{II}}(z) = A_1^{\mathrm{II}}\left(\cosh\frac{\lambda_1}{R}z + \cos\frac{\lambda_1}{R}z\right) + A_2^{\mathrm{II}} \sinh\frac{\lambda_1}{R}z + A_4^{\mathrm{II}} \sin\frac{\lambda_1}{R}z \tag{58}$$

$$\Phi_{\mathrm{II}}(z) = -K_2\left(A_5^{\mathrm{II}} \cosh\frac{\lambda_2}{R}z + A_6^{\mathrm{II}} \sinh\frac{\lambda_2}{R}z\right) - K_3\left(A_7^{\mathrm{II}} \cos\frac{\lambda_3}{R}z + A_8^{\mathrm{II}} \sin\frac{\lambda_3}{R}z\right) - K_4\left(A_9^{\mathrm{II}} \cos\frac{\lambda_4}{R}z + A_{10}^{\mathrm{II}} \sin\frac{\lambda_4}{R}z\right)$$

For tank III

$$X_{\mathrm{III}}(z) = X_{\mathrm{I}}(z)$$

$$Y_{\mathrm{III}}(z) = A_5^{\mathrm{III}} \cosh\frac{\lambda_2}{R}z + A_6^{\mathrm{III}} \sinh\frac{\lambda_2}{R}z + A_7^{\mathrm{III}} \cos\frac{\lambda_3}{R}z + A_8^{\mathrm{III}} \sin\frac{\lambda_3}{R}z + A_9^{\mathrm{III}} \cos\frac{\lambda_4}{R}z + A_{10}^{\mathrm{III}} \sin\frac{\lambda_4}{R}z \tag{59}$$

$$\Phi_{\mathrm{III}}(z) = -K_2\left(A_5^{\mathrm{III}} \cosh\frac{\lambda_2}{R}z + A_6^{\mathrm{III}} \sinh\frac{\lambda_2}{R}z\right) - K_3\left(A_7^{\mathrm{III}} \cos\frac{\lambda_3}{R}z + A_8^{\mathrm{III}} \sin\frac{\lambda_3}{R}z\right) - K_4\left(A_9^{\mathrm{III}} \cos\frac{\lambda_4}{R}z + A_{10}^{\mathrm{III}} \sin\frac{\lambda_4}{R}z\right)$$

For tank IV

$$X_{\mathrm{IV}}(z) = A_5^{\mathrm{IV}} \cosh\frac{\lambda_2}{R}z + A_6^{\mathrm{IV}} \sinh\frac{\lambda_2}{R}z + A_7^{\mathrm{IV}} \cos\frac{\lambda_3}{R}z + A_8^{\mathrm{IV}} \sin\frac{\lambda_3}{R}z + A_9^{\mathrm{IV}} \cos\frac{\lambda_4}{R}z + A_{10}^{\mathrm{IV}} \sin\frac{\lambda_4}{R}z$$

$$Y_{\mathrm{IV}}(z) = Y_{\mathrm{II}}(z) \tag{60}$$

$$\Phi_{\mathrm{IV}}(z) = K_2\left(A_5^{\mathrm{IV}} \cosh\frac{\lambda_2}{R}z + A_6^{\mathrm{IV}} \sinh\frac{\lambda_2}{R}z\right) + K_3\left(A_7^{\mathrm{IV}} \cos\frac{\lambda_3}{R}z + A_8^{\mathrm{IV}} \sin\frac{\lambda_3}{R}z\right) + K_4\left(A_9^{\mathrm{IV}} \cos\frac{\lambda_4}{R}z + A_{10}^{\mathrm{IV}} \sin\frac{\lambda_4}{R}z\right)$$

By taking the linearly independent equations of the original system after the reduction, one forms a 30×30 determinant Δ of the coefficients of the unknowns, which for nontrivial solution must be zero. The determinant is a complicated function of $P = R^4a^2\omega^2$. It should be recalled that parameters K_i, λ_i, and μ_i, where $i = 1, 2, 3$, and 4, are functions of P and therefore are functions of ω. These parameters are given by Eq. (24), the roots of Eq. (30), and Eqs. (39, 42, and 47). Thus the frequency equation is

$$\Delta = 0 \tag{61}$$

The roots of Eq. (61), i.e., the eigenfrequencies, cannot be expressed in closed form. However, by plotting the value of Δ vs P, the eigenfrequencies can be found corresponding to the zeros of Δ. All the functions entering into the determinant are transcendental or trigonometric functions of the roots of Eq. (28). To find the dependence of these roots on $P = R^4a^2\omega^2$, Eq. (28) has been solved by using an IBM computer for different values of P between 0 and 0.12. This covers the spectrum of frequencies with which this paper is concerned.

Equation (28) has two more parameters, m and b. For the present purpose, m was taken as corresponding to an Atlas booster completely filled with liquid fuel, i.e., $m = 0.0667$. Since its value is small, small deviations are not expected to affect the solution. The influence of the parameter b is more interesting. According to Eq. (9), b is the ratio of the bending to the torsional stiffness of each tank. If the material of the tank is isotropic, $b = (1 + \nu)$, where ν is Poisson's ratio. For a nonisotropic tank, however, this ratio can be found experimentally. To cover the whole range of the expected values of b, the roots of Eq. (28) were found for the foregoing range of P and for b = 1.0, 1.1, 1.2, 1.3, 1.4, and 1.5. Such values of the three roots λ_2, λ_3, and λ_4 are plotted in Figs. 3–8, whereas the root λ_1 is given in closed form by Eq. (24). For reasons that will be explained below, the curve λ_3 has been replotted in Fig. 9 on a larger scale for $b = 1.3$. It has been found that such curves, for values of b ranging from $b = 1.0$ to $b = 1.5$. almost coincide. These curves are very useful for graphically finding certain interesting simple modes.

3.3 Obvious Roots of the Determinant

An examination of the characteristic determinant, Eq. (61), shows the following interesting roots without solving the corresponding equations:

$$\mu_3 = k\pi \qquad \lambda_3 = (R/l)k\pi \tag{62}$$

where $k = 1, 2, 3, \ldots$. By adding column A_8^{II} to column A_8^{I} (such an addition does not change the value of the determinant) and column A_8^{IV} to column A_8^{III}, column A_8^{I} becomes equal to column A_8^{III}. Therefore, for $\mu_3 = k\pi$, the determinant is zero, i.e., the value of P corresponding to $\lambda_3 = (R/l)k\pi$ is a root of the determinant (Figs. 5–9). If one also adds column A_8^{I} to column A_8^{II} and column A_8^{III} to column A_8^{IV}, columns A_8^{II} and A_8^{IV} become identical. Therefore, the value of P which corresponds to $\lambda_3 = (R/l)k\pi$ is a *double* root of the characteristic determinant.

The corresponding modes can be found as follows. The homogeneous equations corresponding to all the rows of the characteristic determinant, except rows 19 and 20, form a 28 × 28 system if one introduces the sums $A_8^{\text{I}} + A_8^{\text{III}}$ and $A_8^{\text{II}} + A_8^{\text{IV}}$ as unknowns. This is possible, since the coefficients of A_8^{I} are equal to those of A_8^{III} and the coefficients of A_8^{II} are equal to those of A_8^{IV}. Since the resulting 28 × 28 system has, in general, a nonzero determinant, the only solution is the zero solution. Thus

$$A_i^{\text{I}} = A_i^{\text{II}} = A_i^{\text{III}} = A_i^{\text{IV}} = 0 \; i \neq 0 \tag{63}$$

and

$$A_8^{\text{I}} + A_8^{\text{III}} = 0 \tag{64}$$
$$A_8^{\text{II}} + A_8^{\text{IV}} = 0$$

Equation (64) can be written as

$$A_8^{\text{I}} = -A_8^{\text{III}} = C_1 \tag{65}$$
$$A_8^{\text{II}} = -A_8^{\text{IV}} = C_2$$

Substituting Eqs. (63) and (65) into Eqs. (19) and (20), respectively, of the original system, one obtains

$$2K_3\lambda_3C_1 - 2K_3\lambda_3C_2 = 0 \tag{66}$$

$$-2K_3\lambda_3C_1 + 2K_3\lambda_3C_2 = 0 \tag{67}$$

From Eqs. (66) and (68) it is obvious that $C_1 = C_2 = C$, say, where C equals some arbitrary amplitude. Thus, the modes corresponding to $\mu_3 = k\pi$ are [refer to Eqs. (57–60)]

$$\begin{aligned}
X_{\text{I}} &= 0 & Y_{\text{I}} &= C\sin\frac{k\pi z}{l} \\
X_{\text{II}} &= C\sin\frac{k\pi z}{l} & Y_{\text{II}} &= 0 \\
X_{\text{III}} &= 0 & Y_{\text{III}} &= -C\sin\frac{k\pi z}{l} \\
X_{\text{IV}} &= -C\sin\frac{k\pi z}{l} & Y_{\text{IV}} &= 0
\end{aligned} \tag{68}$$

$$\begin{aligned}
\Phi_{\text{I}} &= K_3C\sin\frac{k\pi z}{l} \\
\Phi_{\text{II}} &= -K_3C\sin\frac{k\pi z}{l} \\
\Phi_{\text{III}} &= K_3C\sin\frac{k\pi z}{l} \\
\Phi_{\text{IV}} &= -K_3C\sin\frac{k\pi z}{l}
\end{aligned}$$

where $K_3 = (\lambda_3^4/P) - 1$ is evaluated for $\lambda_3 = (R/l)k\pi$ and for P corresponding to the values of λ_3 as depicted in the λ_3 curve of Figs. 3–9. By similar reasoning, one can prove that P corresponding to

$$\mu_4 = k\pi \qquad \lambda_4 = (R/l)k\pi \tag{69}$$

where $k = 1, 2, \ldots$, is another *double* root of the characteristic determinant.

The corresponding modes are

$$\begin{aligned}
X_{\text{I}} &= 0 & Y_{\text{I}} &= D\sin\frac{k\pi z}{l} \\
X_{\text{II}} &= D\sin\frac{k\pi z}{l} & Y_{\text{II}} &= 0 \\
X_{\text{III}} &= 0 & Y_{\text{III}} &= -D\sin\frac{k\pi z}{l} \\
X_{\text{IV}} &= -D\sin\frac{k\pi z}{l} & Y_{\text{IV}} &= 0 \\
\Phi_{\text{I}} &= K_4D\sin\frac{k\pi z}{l} \\
\Phi_{\text{II}} &= -K_4D\sin\frac{k\pi z}{l} \\
\Phi_{\text{III}} &= K_4D\sin\frac{k\pi z}{l} \\
\Phi_{\text{IV}} &= -K_4D\sin\frac{k\pi z}{l}
\end{aligned} \tag{70}$$

where D equals an undetermined amplitude, and $K_4 = (\lambda_4^4/P) - 1$ is evaluated for $\lambda_4 = (R/l)k\pi$ and for P corresponding to the value of λ_4 depicted in Figs. 3–8. The values of P corresponding to the three roots of Eq. (30) have been plotted in Figs. 3–8 for various values of b. Since the

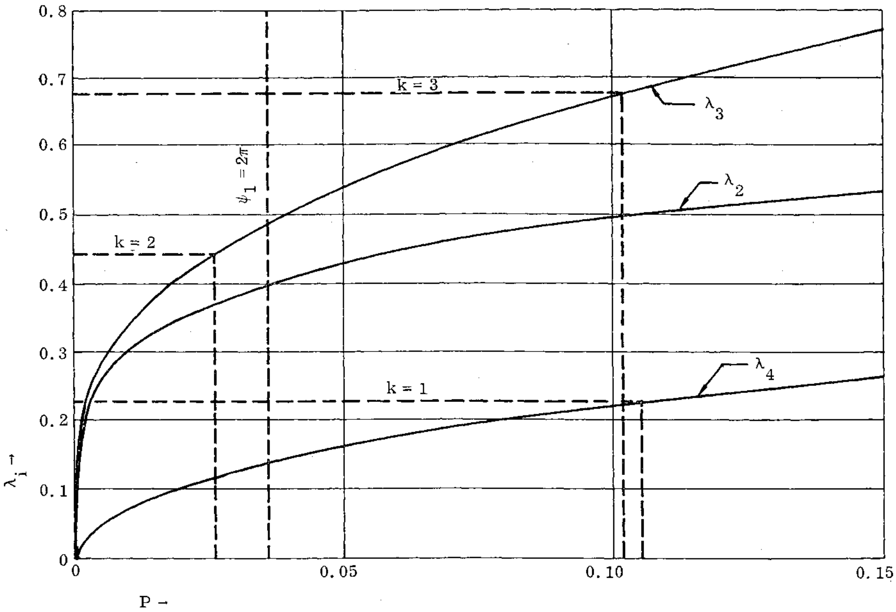

Fig. 3 λ_i vs P, b = 1.0, i = 2, 3, 4

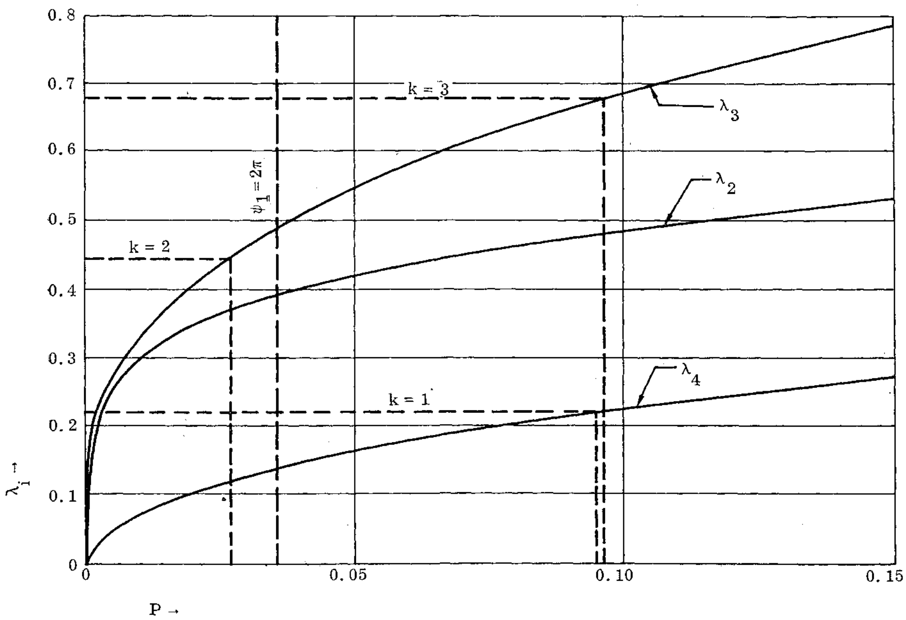

Fig. 4 λ_i vs P, b = 1.1, i = 2, 3, 4

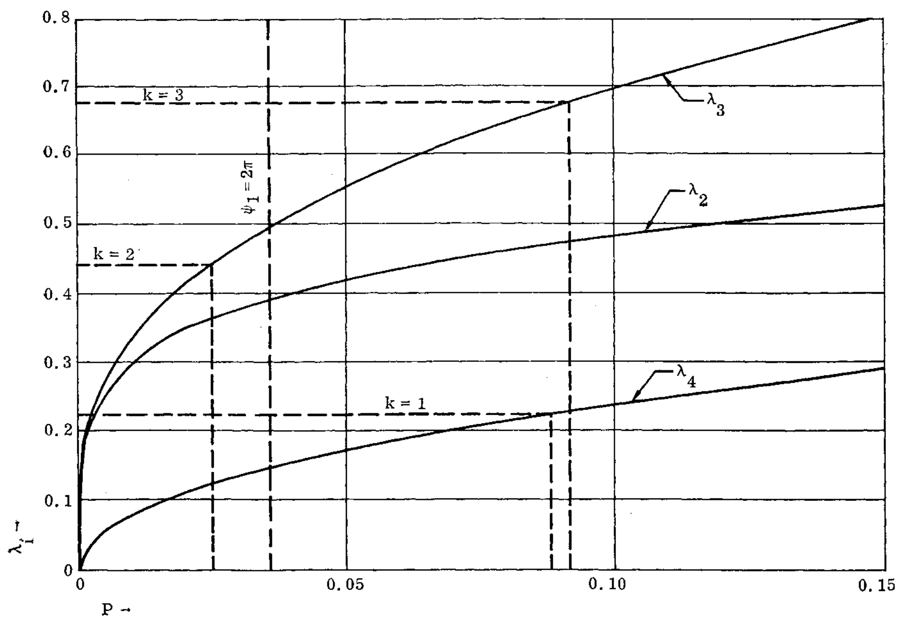

Fig. 5 λ_i vs P, b = 1.2, i = 2, 3, 4

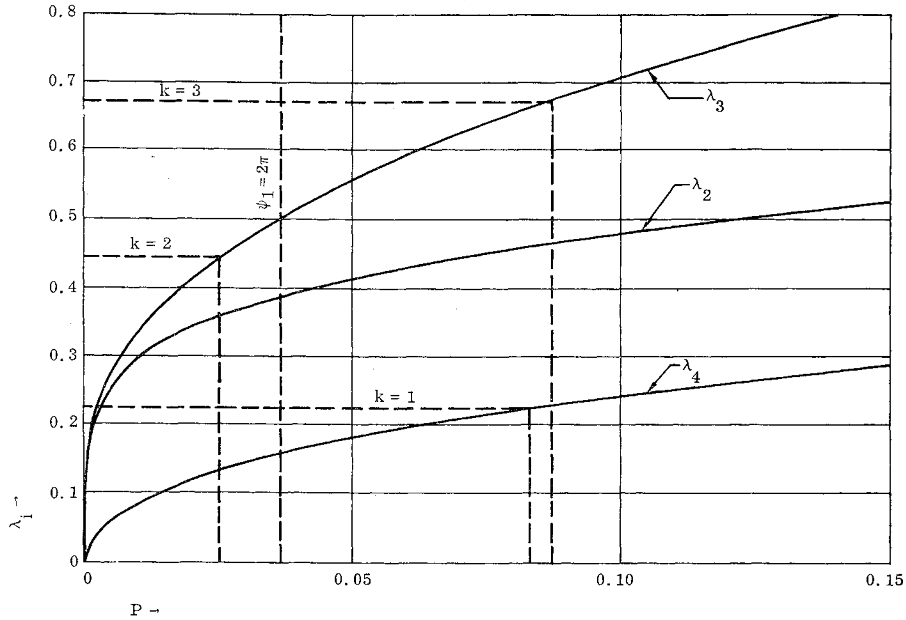

Fig. 6 λ_i vs P, $b = 1.3$, $i = 2, 3, 4$

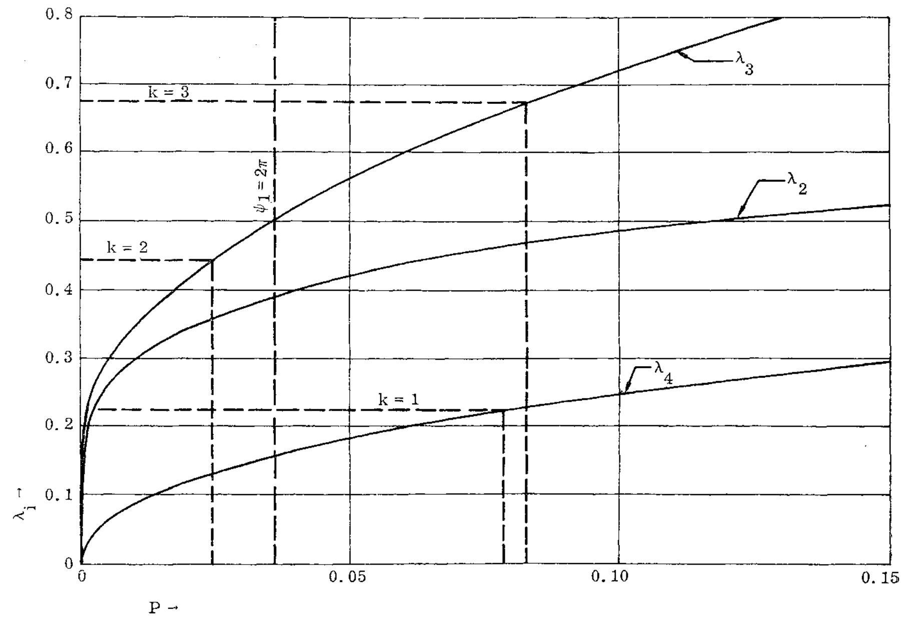

Fig. 7 λ_i vs P, $b = 1.4$, $i = 2, 3, 4$

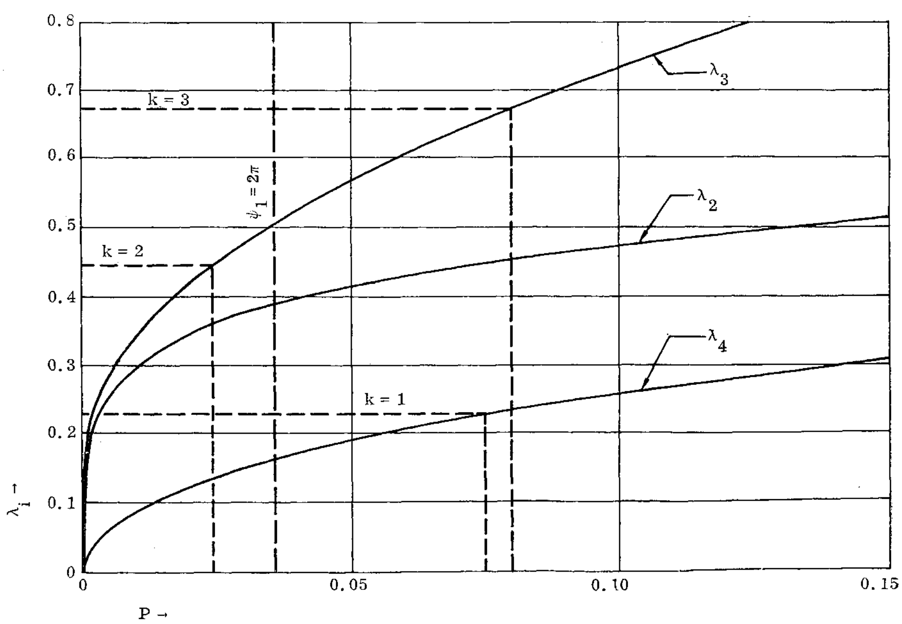

Fig. 8 λ_i vs P, $b = 1.5$, $i = 2, 3, 4$

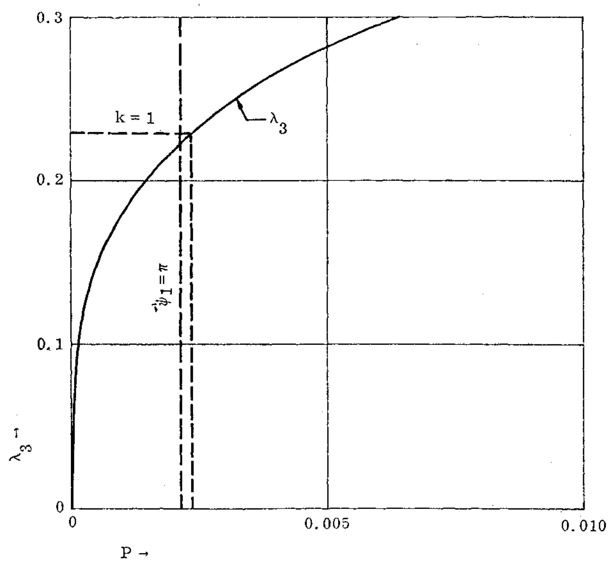

Fig. 9 λ_3 vs P, $b = 1.3$

value of $\mu_3 = \pi$ cannot be determined accurately from these curves, the λ_3 curve for $b = 1.3$ has been plotted in Fig. 9 using a larger scale. This part of the curve does not change appreciably for varying b. In Figs. 3–9 the first eigenfrequencies are also shown, i.e., the value of P corresponding to μ_3 or $\mu_4 = k\pi$ for $k = 1$, 2, and 3, and for $R/l = 14$ (typical of the Atlas). It is worth noting that the value of P corresponding to $\mu_3 = 3\pi$ lies very near that value of P corresponding to $\mu_4 = \pi$. This might present some problems in the design of the control systems. For comparison, the eigenfrequency corresponding to $\mu_1 = 2\pi$ is shown in Figs. 3–8 [refer to Eq. (44)], and the eigenfrequency corresponding to $\mu_1 = \pi$ is shown in Fig. 9. The latter lies very near the eigenfrequency corresponding to $\lambda_3 = \pi$.

4. Matrix Analysis

A matrix analysis for the problem of nonuniform tanks partially filled with liquid has been developed by the authors.[1] In each tank, one lumps the masses of different segments at their middle points and assumes that the flexibility varies linearly between nodal points. The sloshing effect is also represented by an equivalent mass-spring system.[2] In order to find the flexibility of the system in relative coordinates, one has to fix the section of one end of tanks I and III against flexural rotation around the x axis, tanks II and III against flexural rotation around the y axis, and all four sections against torsional rotation around the z axis. However, the formulation of flexibility with this picture of end fixity becomes very complicated. To simplify the problem, one first formulates the flexibility of the system by fixing completely the one end section. By using the flexibility of the unassembled structure,[2] one can find the flexibility of the assembled structure.[3] This flexibility then is modified by relaxing the extra constraints, and the true flexibility in relative coordinates thus is found. The separation of the rigid-body displacements then is carried out. This is obviously necessary, since the inertial forces are expressed in absolute displacements through the masses, whereas the elastic forces are expressed in relative displacements through the flexibility. This separation is obtained by writing the Lagrange equations of motion in terms of the relative displacements by transforming the mass matrix accordingly. Thus a characteristic equation is obtained in determinant form from which the eigenfrequencies can be obtained numerically. The modes of vibration thus obtained indicate that, in general, both flexural and torsional modes are excited at the same time.

Conclusion

An analysis of the vibrations of a cluster of four uniform tanks filled with liquid fuel has been presented. The problem of eigenfrequencies has been reduced to finding the roots of a 30×30 determinant. Certain obvious roots, corresponding to simple modes, have been found by inspecting the characteristic determinant. To find other frequencies, a solution of a 30×30 determinant, having transcendental and trigonometric functions of the frequency, generally has to be found by means of a digital computer.

A lumped-parameter matrix approach to the problem has been outlined briefly. A detailed account of this procedure may be found in Ref. 1.

It is believed that the present analysis will find a useful application in the design of control systems for clustered boosters. The simultaneous presence of both bending and torsional modes, the closeness of certain natural frequencies and the double roots of the characteristic determinant, as contrasted to single-tank boosters, provide additional problems for the control designer.

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